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Superconformal boundary conditions for the WZW model

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Abstract: We review the most general, local, superconformal boundary conditions for the two-dimensional $N = 1$ and $N = 2$ non-linear sigma models, and analyse them for the $N = 1$ and $N = 2$ supersymmetric WZW models. We find that the gluing map between the left and right affine currents is generalised in a very specific way as compared to the constant Lie algebra automorphisms that are known.

Keywords: Sigma Models, D-branes, Extended Supersymmetry.

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1. Introduction

In recent years D-branes on group manifolds have attracted a great deal of attention as an ideal setting for the study of D-branes in general string backgrounds. Because the world-sheet theories corresponding to strings on a group manifold are solvable Wess-Zumino-Witten (WZW) conformal field theories, one can analyse the stringy effects in great detail. WZW models have an underlying affine symmetry which corresponds to affine Lie algebras. By now D-branes associated with an automorphism of the affine algebra are fairly well understood, at least in the compact case [1, 2]. Different properties of these branes such as stability [3], underlying geometry [4, 5], effective actions [6]–[9] etc., have been extensively studied.

Despite this progress there is a wide class of D-branes (i.e., boundary conditions) of WZW models which are less well understood and which correspond to the preservation of only (super)conformal symmetry. The only models which are rational with respect to (super)conformal symmetry are the (super) minimal models. The compact WZW models are rational only with respect to the affine symmetry but not with respect to (super)conformal symmetry. Therefore it is not so straightforward to analyse the (super)conformal branes. Interesting steps in this direction have been taken in [10].

In the present letter we would like to attack the problem from a different side. The WZW model provides a typical example of a string background admitting a sigma model description. This allows one to undertake a complementary study of the possible D-brane configurations. The sigma model description is valuable since it provides a geometric interpretation of D-branes in the semiclassical limit. We would like to analyse classical superconformal conditions that are local in terms of the sigma model fields and that do have such an interpretation. Furthermore we want to rewrite and interpret these boundary conditions in terms of gluing conditions for the affine currents. At the classical level, conformal symmetry is too weak a requirement on its own to draw any conclusions. Besides, conformal symmetry by itself does not guarantee a consistent geometrical description. As has been shown in [11, 12], one must impose minimal supersymmetry together with conformality to obtain boundary conditions with a consistent geometrical description in terms of submanifolds.

The paper is organised as follows. We begin by reviewing the most general local superconformal boundary conditions of the non-linear sigma model. In section 2 we review the results obtained in [11, 12] for $N=1$, and in section 3 we review the $N=2$ conditions derived in [13, 14]. These conditions are then analysed for the special cases of $N=1$ and $N=2$ WZW models, in sections 4 and 5, respectively. Finally, in section 6, we summarise and discuss the conclusions.

2. $N=1$ review

Here we give a summary of the derivation of boundary conditions for the $N=1$ supersymmetric non-linear sigma model; for details, see [11, 12]. This model is given by the following action,

$$S=\int_\Sigma d^2\xi\left[\partial_{\#}X^\mu\partial_{=}X^\nu E_{\mu\nu}+i\psi_+^\mu\nabla_-^{(+)}\psi_+^\nu g_{\mu\nu}+i\psi_-^\mu\nabla_+^{(-)}\psi_-^\nu g_{\mu\nu}+\frac{1}{2}\psi_+^\lambda\psi_+^\sigma\psi_-^\rho\psi_-^\gamma\mathcal{R}^-_{\rho\gamma\lambda\sigma}\right], \tag{2.1}$$

where $E_{\mu\nu}\equiv g_{\mu\nu}+B_{\mu\nu}$ denotes a background of general (riemannian) metric $g_{\mu\nu}$ and two-form B with field strength $H=dB$. The curvature $\mathcal{R}^\pm_{\rho\gamma\lambda\sigma}$ is defined as

$$\mathcal{R}^{\pm\mu}{}_{\sigma\rho\lambda}=\Gamma^{\pm\mu}{}_{\lambda\sigma,\rho}-\Gamma^{\pm\mu}{}_{\rho\sigma,\lambda}+\Gamma^{\pm\mu}{}_{\rho\gamma}\Gamma^{\pm\gamma}{}_{\lambda\sigma}-\Gamma^{\pm\mu}{}_{\lambda\gamma}\Gamma^{\pm\gamma}{}_{\rho\sigma}\,, \tag{2.2}$$

with $\Gamma^\pm$ given by

$$\Gamma^{\pm\nu}{}_{\rho\sigma}\equiv\Gamma^\nu{}_{\rho\sigma}\pm g^{\nu\mu}H_{\mu\rho\sigma}\,, \tag{2.3}$$

and $\nabla_\pm^{(\pm)}$ by

$$\nabla_\pm^{(+)}\psi_+^\nu=\partial_{\substack{+\\=}}\psi_+^\nu+\Gamma^{+\nu}{}_{\rho\sigma}\partial_{\substack{+\\=}}X^\rho\psi_+^\sigma\,,\qquad\nabla_\pm^{(-)}\psi_-^\nu=\partial_{\substack{+\\=}}\psi_-^\nu+\Gamma^{-\nu}{}_{\rho\sigma}\partial_{\substack{+\\=}}X^\rho\psi_-^\sigma\,. \tag{2.4}$$

At the classical level the model (2.1) has $N=(1,1)$ superconformal symmetry modulo boundary terms. This symmetry gives rise to conserved currents, namely the stress tensor T and the supersymmetry current G^1. The components of the currents are

$$G^1_+ = \psi^\mu_+ \partial_{+\!\!\!+} X^\nu g_{\mu\nu} - \frac{i}{3}\psi^\mu_+\psi^\nu_+\psi^\rho_+ H_{\mu\nu\rho}\,, \tag{2.5}$$

$$G^1_- = \psi^\mu_- \partial_{=} X^\nu g_{\mu\nu} + \frac{i}{3}\psi^\mu_-\psi^\nu_-\psi^\rho_- H_{\mu\nu\rho}\,, \tag{2.6}$$

$$T_{++} = g_{\mu\nu}\partial_{+\!\!\!+} X^\mu \partial_{+\!\!\!+} X^\nu + i g_{\mu\nu}\psi^\mu_+ \nabla^{(+)}_+ \psi^\nu_+\,, \tag{2.7}$$

$$T_{--} = g_{\mu\nu}\partial_{=} X^\mu \partial_{=} X^\nu + i g_{\mu\nu}\psi^\mu_- \nabla^{(-)}_- \psi^\nu_-\,, \tag{2.8}$$

and they are conserved, $\partial_{\substack{= \\ +\!\!\!+}} G^1_\mp = 0$ and $\partial_{\substack{= \\ +\!\!\!+}} T_{\pm\pm} = 0$. In the absence of a boundary, these conservation laws give rise to four conserved charges. However, in the presence of boundaries one must take extra care when deriving the conserved charges from $T_{\pm\pm}$ and $G^1_\pm$. It turns out that the appropriate boundary conditions to be imposed on the currents are

$$[T_{++} - T_{--}]_{\sigma=0,\pi} = 0\,, \qquad \left[G^1_+ - \eta_1 G^1_-\right]_{\sigma=0,\pi} = 0\,, \tag{2.9}$$

where the world-sheet Σ is assumed to be $\mathbb{R}\times[0,\pi]$ (i.e., $\tau\in\mathbb{R}$ and $\sigma\in[0,\pi]$), and $\eta_1=\pm1$ corresponds to the choice of spin structures. Our goal is to solve conditions (2.9) in terms of X^μ and $\psi^\mu_\pm$ such that the solution is local in these fields (the locality is important for a geometrical interpretation of the boundary conditions). The most general local fermionic boundary condition allowed by dimensional analysis has the simple form

$$\psi^\mu_- = \eta_1 R^\mu{}_\nu(X)\psi^\nu_+\,, \tag{2.10}$$

where $R^\mu{}_\nu$(X) is a locally defined object which transforms as a (1,1) tensor field under coordinate transformations. The bosonic counterpart of (2.10) can be derived by means of a supersymmetry transformation,[1] and reads

$$\partial_= X^\mu - R^\mu{}_\nu \partial_{+\!\!\!+} X^\nu + 2i(P^\sigma{}_\rho \nabla_\sigma R^\mu{}_\nu + P^\mu{}_\gamma g^{\gamma\delta} H_{\delta\sigma\rho} R^\sigma{}_\nu)\psi^\rho_+\psi^\nu_+ = 0\,, \tag{2.11}$$

where $P^\mu{}_\nu \equiv (\delta^\mu{}_\nu + R^\mu{}_\nu)/2$.

In [12] we have shown that the expressions (2.10) and (2.11) are the solutions of (2.9) provided that $R^\mu{}_\nu$ satisfies the following conditions,

$$g_{\rho\sigma} = R^\mu{}_\rho g_{\mu\nu} R^\nu{}_\sigma\,, \tag{2.12}$$

$$P^\rho{}_\tau R^\mu{}_\sigma g_{\mu\nu}\nabla_\rho R^\nu{}_\gamma + P^\rho{}_\sigma R^\mu{}_\gamma g_{\mu\nu}\nabla_\rho R^\nu{}_\tau + P^\rho{}_\gamma R^\mu{}_\tau g_{\mu\nu}\nabla_\rho R^\nu{}_\sigma + 4P^\mu{}_\tau P^\nu{}_\sigma P^\rho{}_\gamma H_{\mu\nu\rho} = 0\,. \tag{2.13}$$

To understand the geometrical meaning of these conditions we need to introduce new objects. We define a projector $Q^\mu{}_\nu$ (i.e., $Q^2=Q$) such that $RQ=QR=-Q$. The complementary projector $\pi = I - Q$ (I is the identity operator) satisfies $\pi P = P\pi = P$. Then, by contracting (2.13) with $Q^\tau{}_\lambda$, we find the condition

$$\pi^\mu{}_\gamma \pi^\rho{}_\nu Q^\delta{}_{[\mu,\rho]} = 0\,, \tag{2.14}$$

[1]For an explicit component form of this transformation, see appendix A in [12].

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which is the integrability condition for π. Therefore *there is a maximal integral submanifold* which corresponds to the distribution π. We may also contract (2.11) with Q, and use the integrability condition (2.14), to obtain that $\pi^\mu{}_\nu \partial_0 X^\nu = \partial_0 X^\mu$ (or equivalently $Q^\mu{}_\nu \partial_0 X^\nu = 0$) and thus the end of the string is confined in the directions transverse to a given maximal integral submanifold.

The object R^μ_ν carries additional information. We define a B-field[2] $B^\pi_{\mu\nu}$, i.e., a two-form living on the submanifold corresponding to π, by

$$\pi^\rho{}_\mu g_{\rho\sigma} \pi^\sigma{}_\nu - B^\pi_{\mu\nu} = \left(\pi^\rho{}_\mu g_{\rho\sigma} \pi^\sigma{}_\lambda + B^\pi_{\mu\lambda}\right) R^\lambda{}_\nu \,. \tag{2.15}$$

Contracting (2.13) with π and using the definition (2.15) we find that

$$(dB^\pi)_{\mu\nu\rho} = \pi^\lambda{}_\mu \pi^\gamma{}_\nu \pi^\sigma{}_\rho H_{\lambda\gamma\sigma} \,. \tag{2.16}$$

Thus an R^μ_ν which satisfies (2.13) encodes both the integrable distribution π and the two-form $B^\pi_{\mu\nu}$ living on the corresponding integral submanifold. In addition dB^π coincides with the pull-back of $H_{\mu\nu\rho}$ to this submanifold. For details of the analysis leading to the above results we refer the reader to [11] and [12].

In summary, there is a one-to-one correspondence between local superconformal boundary conditions and submanifolds with extra (B-field) structure.

3. $N = 2$ review

We now turn to the $N = 2$ sigma model, reviewing the results of [13].

Any riemannian target manifold $\mathcal{M}$ admits an $N = (1,1)$ sigma model. If the geometry on $\mathcal{M}$ is further restricted [15] the sigma model can have $N = (2,2)$ supersymmetry. The target manifold $\mathcal{M}$ of the $N = (2,2)$ sigma model must be equipped with two complex structures $J^\mu_{\pm\nu}$, and the metric g must be hermitean with respect to both of these. These complex structures should be such that

$$l^{\mu\rho} \equiv (J^\mu_{+\nu} + J^\mu_{-\nu}) g^{\nu\rho} \,, \qquad m^{\mu\rho} \equiv (J^\mu_{+\nu} - J^\mu_{-\nu}) g^{\nu\rho} \tag{3.1}$$

are Poisson bi-vectors. The torsion $H_{\mu\nu\rho}$ is then defined via the Schouten bracket of these two Poisson structures [16].

In addition to the currents (2.5)–(2.8), an $N = (2,2)$ sigma model has two further supersymmetry currents, $G^2_\pm$, and two U(1) R-symmetry currents $J_\pm$. In terms of world-sheet fields the currents read

$$G^2_+ = \psi^\mu_+ \partial_{\#} X^\nu J_{+\mu\nu} + \frac{i}{3} \psi^\mu_+ \psi^\nu_+ \psi^\rho_+ J^\lambda_{+\mu} J^\sigma_{+\nu} J^\gamma_{+\rho} H_{\lambda\sigma\gamma} \,, \tag{3.2}$$

$$G^2_- = \psi^\mu_- \partial_{=} X^\nu J_{-\mu\nu} - \frac{i}{3} \psi^\mu_- \psi^\nu_- \psi^\rho_- J^\lambda_{-\mu} J^\sigma_{-\nu} J^\gamma_{-\rho} H_{\lambda\sigma\gamma} \,, \tag{3.3}$$

$$J_+ = \psi^\mu_+ \psi^\nu_+ J_{+\mu\nu} \,, \tag{3.4}$$

$$J_- = \psi^\mu_- \psi^\nu_- J_{-\mu\nu} \,. \tag{3.5}$$

[2] More precisely, a gauge invariant combination of a U(1)-field strength and the background B-field $B_{\mu\nu}$.

If we want to preserve the maximal possible amount of the bulk supersymmetries in the presence of a boundary, then in addition to the conditions (2.9) we need to simultaneously require the following boundary conditions for the currents $G^2_\pm$ and $J_\pm$,

$$[G^2_+ - \eta_2 G^2_-]_{\sigma=0,\pi} = 0\,, \qquad [J_+ - (\eta_1\eta_2)J_-]_{\sigma=0,\pi} = 0\,, \tag{3.6}$$

where $\eta_2 = \pm 1$. As a result, the conditions (2.12) and (2.13) on $R^\mu_{\ \nu}$ must be supplemented with the following,

$$J^\mu_{-\lambda} R^\lambda_{\ \nu} = (\eta_1\eta_2) R^\mu_{\ \lambda} J^\lambda_{+\nu}\,, \tag{3.7}$$

$$J^\lambda_{+[\mu} H_{|\lambda|\nu|\sigma|} P^\sigma_{\ \rho]} + J^\lambda_{+[\mu} R^\sigma_{\ |\lambda|} P^\gamma_{\ \nu} R^\delta_{\ \rho]} H_{\sigma\gamma\delta} - J^\lambda_{+[\mu} R^\phi_{\ |\lambda} \eta_{\phi\gamma} \nabla_{\sigma|} R^\gamma_{\ \rho} P^\sigma_{\ \nu]} = 0, \tag{3.8}$$

where the case $(\eta_1\eta_2) = 1$ corresponds to B-type and $(\eta_1\eta_2) = -1$ to A-type models.

It was shown in [13] that (3.8) is automatically satisfied, given the other three conditions, (2.12), (2.13) and (3.7). Nevertheless, it is interesting to note that (3.8) corresponds to the invariance of the two-fermion term in (2.11) under the appropriate combination of U(1)-symmetries.

Geometrically the $N = 2$ superconformal boundary conditions should be interpreted in terms of submanifolds of special types. However, the problem has not been solved in all generality (some observations on the subject are made in [13]). Only $N = 2$ D-branes on Kähler manifolds (i.e., $J^\mu_{+\nu} = \pm J^\mu_{-\nu}$) are well understood in terms of symplectic geometry. Unfortunately this case is irrelevant to the WZW models since group manifolds are never Kähler.

4. The $N = 1$ WZW model

We now analyse the $N = 1$ boundary conditions given in section 2 for the WZW model. We begin by reviewing some basics pertaining to WZW models, in the process introducing some notation that will be useful in the analysis.

4.1 Preliminaries

The WZW models represent a special class of non-linear sigma models defined over a group manifold $\mathcal{M}$ of some Lie group $\mathcal{G}$. The isometry group $\mathcal{G}\times\mathcal{G}$ is generated by the left- and right-invariant Killing vectors l^μ_A and r^μ_A respectively, where $A = 1, 2, \dots, \dim\mathcal{G}$. They satisfy

$$\{l_A, l_B\} = f_{AB}{}^C l_C\,, \qquad \{r_A, r_B\} = -f_{AB}{}^C r_C\,, \qquad \{l_A, r_B\} = 0\,, \tag{4.1}$$

where $\{\,\cdot\,,\,\cdot\,\}$ is the Lie bracket for vector fields. We restrict ourselves to semi-simple Lie groups, so that the Cartan-Killing metric η_{AB} has an inverse η^{AB} and can be used to raise and lower Lie algebra indices. Both l^μ_A and r^μ_A can be regarded as vielbeins, with inverses l^A_μ and r^A_μ, respectively. To define the sigma model, we choose the invariant metric

$$g_{\mu\nu} = \frac{1}{\rho^2}\, l^A_\mu l^B_\nu \eta_{AB} = \frac{1}{\rho^2}\, r^A_\mu r^B_\nu \eta_{AB}\,, \tag{4.2}$$

while $H_{\mu\nu\rho}$ is proportional to the structure constants of the corresponding Lie algebra $\mathbf{g}$,

$$H_{\mu\nu\rho} = \frac{1}{2} k\, l_\mu^A l_\nu^B l_\rho^C f_{ABC} = \frac{1}{2} k\, r_\mu^A r_\nu^B r_\rho^C f_{ABC}\,. \tag{4.3}$$

Here ρ and k are constants, and k must satisfy a quantization condition. If $\rho^2 = \pm 1/k$, then $H_{\mu\nu\rho}$ is the parallelising torsion on the group manifold and this is also precisely the relation between the coupling constants that holds at the conformal fixed point of the beta-functions. Since we are interested in the conformal model, we set $\rho^2 = 1/k$ in the following discussion. Moreover, since k appears only as an overall factor in our calculations, we may set $k = 1$.

We thus study the sigma model (2.1) with $g_{\mu\nu}$ and $H_{\mu\nu\rho}$ given by (4.2) and (4.3). From the above properties follows that the left- and right-invariant Killing vectors satisfy

$$\nabla_\rho^{(-)} l_A^\mu = 0\,, \qquad \nabla_\rho^{(+)} r_A^\mu = 0\,, \tag{4.4}$$

where $\nabla_\rho^{(\pm)}$ are the affine connections defined in (2.3). The relations (4.4) are the Cartan-Maurer equations for our group manifold. They imply the existence of chiral (antichiral) Lie algebra valued currents,

$$\mathcal{J}_-^A = l_\mu^A D_- \Phi^\mu\,, \qquad \mathcal{J}_+^A = -r_\mu^A D_+ \Phi^\mu\,, \tag{4.5}$$

obeying $D_\mp \mathcal{J}_\pm^A = 0$. The components of these currents are defined as

$$j_\pm^A = \mathcal{J}_\pm^A|\,, \qquad k^A_{\substack{+\!\!\!+\\ =}} = -iD_\pm \mathcal{J}_\pm^A|\,. \tag{4.6}$$

It is important to notice that there are two different sets of bosonic affine currents, $\mathcal{K}^A_{\substack{+\!\!\!+\\ =}}$ and $k^A_{\substack{+\!\!\!+\\ =}}$, related to each other by

$$\begin{aligned} k_=^A &= \mathcal{K}_=^A + \frac{i}{2} f_{BC}{}^A j_-^B j_-^C\,, \\ k_{+\!\!\!+}^A &= \mathcal{K}_{+\!\!\!+}^A + \frac{i}{2} f_{BC}{}^A j_+^B j_+^C\,, \end{aligned} \tag{4.7}$$

where

$$\mathcal{K}_=^A \equiv l_\mu^A \partial_= X^\mu\,, \qquad \mathcal{K}_{+\!\!\!+}^A \equiv -r_\mu^A \partial_{+\!\!\!+} X^\mu\,. \tag{4.8}$$

In terms of group elements g, they correspond to the currents $\mathcal{K}_= = g^{-1}dg$ and $\mathcal{K}_{+\!\!\!+} = dg\, g^{-1}$. However, here we will stick to the coordinate representation.

Note that, on a group manifold with metric (4.2) and torsion (4.3), with $\rho^2 = 1/k$, the four-fermion term in the action (2.1) vanishes. Using the Lie algebra valued fermion fields defined in (4.6),

$$j_-^A = l_\mu^A \psi_-^\mu\,, \qquad j_+^A = -r_\mu^A \psi_+^\mu\,, \tag{4.9}$$

we can rewrite (2.1) as

$$S = \int d^2\xi \left[\partial_{+\!\!\!+} X^\mu \partial_= X^\nu E_{\mu\nu} + i\, j_-^A \eta_{AB}\, \partial_{+\!\!\!+} j_-^B + i\, j_+^A \eta_{AB}\, \partial_= j_+^B\right], \tag{4.10}$$

where the bosonic part can be written in terms of the group elements as usual.

4.2 Boundary conditions

Here we discuss the $N = 1$ superconformal boundary conditions which we reviewed in section 2, for the WZW models. The conditions (2.10) and (2.11) can be rewritten in terms of the affine bosonic and fermionic currents, as

$$j_-^A = \eta_1 R^A{}_B j_+^B \,, \tag{4.11}$$

$$\mathcal{K}_=^A - R^A{}_B \mathcal{K}_{+\!\!+}^B + \tfrac{i}{2}\left(f_{SM}{}^A R^S{}_D R^M_L - f_{DL}{}^C R^A{}_C - 2\mathcal{L}_D R^A{}_L\right) j_+^D j_+^L = 0\,. \tag{4.12}$$

Here we have defined

$$R^\mu{}_\nu \equiv -l^\mu_A R^A{}_B r^B_\nu \,, \qquad \mathcal{L}_D R^A{}_L \equiv k^\mu_D \partial_\mu R^A{}_L \,, \qquad k^\mu_A \equiv r^\mu_A - R^L{}_A l^\mu_L \,. \tag{4.13}$$

Thus $R^A{}_B : \mathcal{G} \to \mathbf{g} \otimes \mathbf{g}^*$ is a map from the group to the tensor product of the Lie algebra with its dual.

In terms of Lie algebra quantities, the conditions (2.12) and (2.13) can similarly be rewritten as

$$R^C{}_A \eta_{CD} R^D{}_B = \eta_{AB} \,, \tag{4.14}$$

$$f_{ABC} - R^D{}_A R^L{}_B R^M_C f_{DLM} = -\eta_{LM} R^M_{[C} \mathcal{L}_A R^L{}_{B]} \,. \tag{4.15}$$

The condition (4.15) is a first-order differential equation for the gluing matrix $R^A{}_B$. The most obvious solution of eq. (4.15) is a constant matrix $R^A{}_B$, i.e., $\mathcal{L}_C R^A{}_B = 0$. In this case $R^A{}_B$ corresponds to a Lie algebra automorphism of $\mathbf{g}$. The boundary conditions (4.11) and (4.12) are then local in terms of the affine currents. This case is best known and geometrically corresponds to (twisted) conjugacy classes [4, 5]. Note that (4.15) allows also nonconstant $R^A{}_B$ to be Lie algebra automorphisms, as long as they satisfy $\mathcal{L}_{[A} R^C{}_{B]} = 0$.

Although it is plausible from a technical point of view to write the boundary conditions in a local form for the affine currents, there is no physical reason why all (in some sense) reasonable conditions should be local in these currents. At the level of sigma models, however, locality in terms of X^μ and $\psi^\mu_\pm$ is necessary for a geometrical interpretation.

Another interesting point is that, if we require the two-fermion term in (4.12) to be absent, then combining this with (4.15), we find that $R^A{}_B$ is constant along π, that is, $\pi^\mu{}_\nu \partial_\mu R^A{}_B = 0$. Thus it corresponds to a Lie algebra automorphism along π.

5. The $N = 2$ WZW model

5.1 Preliminaries

The problem of $N = 2$ supersymmetry for WZW models was first addressed in [17, 18, 19]. However, we will not follow the original presentation; instead, the basic approach used here, as well as some of the results, may be found in [14].

The complex structures $J^\mu_{\pm\nu}$ are, respectively, left- and right-invariant, and are of the form

$$J^\mu_{-\nu} = l^\mu_A J^A_B l^B_\nu \,, \qquad J^\mu_{+\nu} = r^\mu_A \tilde{J}^A_B r^B_\nu \,, \tag{5.1}$$

where J^A_B and $\tilde{J}^A_B$ are constant matrices acting on the Lie algebra. Restricting attention to the left-invariant complex structure J^A_B, bulk supersymmetry imposes the following constraints,

$$J^A_C J^C_B = -\delta^A_B \,, \tag{5.2}$$
$$J^C_A \eta_{CD} J^D_B = \eta_{AB} \,, \tag{5.3}$$
$$f_{ABC} = J^D_A J^L_B f_{DLC} + J^D_B J^L_C f_{DLA} + J^D_C J^L_A f_{DLB} . \tag{5.4}$$

Thus we need to construct a J^A_B on the Lie algebra $\mathbf{g}$ satisfying (5.2)–(5.4). This is possible only for even-dimensional Lie algebras. J^A_B has eigenvalues $\pm i$, and we choose a basis $T_A = (T_a, T_{\bar{a}})$ on the Lie algebra $\mathbf{g}$ such that J^A_B is diagonal: $J^a_b = i\delta^a_b$, $J^{\bar{a}}_{\bar{b}} = -i\delta^{\bar{a}}_{\bar{b}}$. In this basis eq. (5.3) leads to $\eta_{ab} = \eta_{\bar{a}\bar{b}} = 0$, and (5.4) gives $f_{abc} = f_{\bar{a}\bar{b}\bar{c}} = 0$. Taken together, this implies that $f_{ab}{}^{\bar{c}} = 0$ and $f_{\bar{a}\bar{b}}{}^{c} = 0$, so that the two sets of generators $\{T_a\}$ and $\{T_{\bar{a}}\}$ form Lie subalgebras of $\mathbf{g}$, call them $\mathbf{g}_+$ and $\mathbf{g}_-$, respectively. These subalgebras are maximally isotropic subspaces with respect to η_{AB}. Thus the complex structures on the even-dimensional group are related to a decomposition of the Lie algebra $\mathbf{g}$ into two maximally isotropic subalgebras with respect to η_{AB}, such that $\mathbf{g} = \mathbf{g}_- \oplus \mathbf{g}_+$ as a vector space. Such a structure is called a Manin triple $(\mathbf{g}, \mathbf{g}_-, \mathbf{g}_+)$, and was initially introduced by Drinfeld in the context of completely integrable systems and quantum groups [20]. The relevance of Manin triples to $N = 2$ supersymmetry on group manifolds was pointed out in [21].

In the general situation the $N = (2, 2)$ WZW model would be equipped with a left Manin triple $(\mathbf{g}, \mathbf{g}_-, \mathbf{g}_+)$, and a right Manin triple $(\mathbf{g}, \tilde{\mathbf{g}}_-, \tilde{\mathbf{g}}_+)$. However, both left and right Manin triples are defined with respect to the same ad-invariant bilinear nondegenerate form η_{AB}. If the left and right Manin triples are the same (i.e., $J^A_B = \tilde{J}^A_B$ modulo a Lie algebra automorphism), then the bivector $(J_+ - J_-)g^{-1}$ defines a Poisson-Lie structure. Here we consider the situation where the left and right Manin triples are the same; for different left and right Manin triples the generalisation is straightforward.

5.2 Boundary conditions

In this section we study the implications of the $N = 2$ boundary conditions (reviewed in section 3) for the WZW model. The currents may be written in terms of Lie algebra quantities (see appendix A), and the boundary conditions corresponding to (3.7) and (3.8) read

$$R^B_A J^C_B = (\eta_1\eta_2) J^B_A R^C_B \,, \tag{5.5}$$
$$J_{NM} R^N_{[A} \mathcal{L}_B R^M_{C]} = 0 \,. \tag{5.6}$$

We first analyse the B-type conditions, $(\eta_1\eta_2) = 1$. In this case eq. (5.5) implies that, in our chosen basis, $R^a_{\bar{b}} = R^{\bar{a}}_b = 0$. Then we find from the $N = 1$ condition (4.15) that (recall that $f_{abc} = f_{\bar{a}\bar{b}\bar{c}} = 0$)

$$f_{ab\bar{c}} - f_{dl\bar{m}} R^d_a R^l_b R^{\bar{m}}_{\bar{c}} = -\eta_{l\bar{m}} R^{\bar{m}}_{\bar{c}} \left(\mathcal{L}_{[a} R^l_{b]} \right) , \tag{5.7}$$
$$f_{\bar{a}\bar{b}c} - f_{\bar{d}\bar{l}m} R^{\bar{d}}_{\bar{a}} R^{\bar{l}}_{\bar{b}} R^m_c = -\eta_{\bar{l}m} R^m_c \left(\mathcal{L}_{[\bar{a}} R^{\bar{l}}_{\bar{b}]} \right) . \tag{5.8}$$

Using (4.14), we can write this as

$$f_{ab}{}^{s}R^{l}{}_{s} - f_{mn}{}^{l}R^{m}{}_{a}R^{n}{}_{b} = -\mathcal{L}_{[a}R^{l}{}_{b]}\,, \tag{5.9}$$

$$f_{\bar{a}\bar{b}}{}^{\bar{s}}R^{\bar{l}}{}_{\bar{s}} - f_{\bar{m}\bar{n}}{}^{\bar{l}}R^{\bar{m}}{}_{\bar{a}}R^{\bar{n}}{}_{\bar{b}} = -\mathcal{L}_{[\bar{a}}R^{\bar{l}}{}_{\bar{b}]}\,. \tag{5.10}$$

For the A-type conditions, $(\eta_1\eta_2) = -1$, eq. (5.5) implies that $R^{a}{}_{b} = R^{\bar{a}}{}_{\bar{b}} = 0$. The relations corresponding to (5.7) and (5.8) for A-type are then

$$f_{ab\bar{c}} - f_{\bar{d}\bar{l}m}R^{\bar{d}}_{a}R^{\bar{l}}_{b}R^{m}_{\bar{c}} = -\eta_{l\bar{m}}R^{l}_{\bar{c}}\left(\mathcal{L}_{[a}R^{\bar{m}}_{b]}\right), \tag{5.11}$$

$$f_{\bar{a}\bar{b}c} - f_{dl\bar{m}}R^{d}{}_{\bar{a}}R^{l}{}_{\bar{b}}R^{\bar{m}}_{c} = -\eta_{l\bar{m}}R^{\bar{m}}_{c}\left(\mathcal{L}_{[\bar{a}}R^{l}{}_{\bar{b}]}\right), \tag{5.12}$$

or equivalently,

$$f_{ab}{}^{s}R^{\bar{l}}{}_{s} - f_{\bar{m}\bar{n}}{}^{\bar{l}}R^{\bar{m}}{}_{a}R^{\bar{n}}{}_{b} = -\mathcal{L}_{[a}R^{\bar{l}}{}_{b]}\,, \tag{5.13}$$

$$f_{\bar{a}\bar{b}}{}^{\bar{s}}R^{l}{}_{\bar{s}} - f_{mn}{}^{l}R^{m}{}_{\bar{a}}R^{n}{}_{\bar{b}} = -\mathcal{L}_{[\bar{a}}R^{l}{}_{\bar{b}]}\,. \tag{5.14}$$

Let us analyse the B-type equation (5.9) in detail (the analogous consideration can be applied to (5.10), as well as to the A-type equations). To understand the geometrical content of this condition, we first consider the vector $k^{\mu}_{a} = r^{\mu}_{a} - R^{l}{}_{a}l^{\mu}_{l}$, with respect to which the Lie derivative in (5.9) is taken. We can ask when k^{μ}_{a} satisfies the Lie algebra bracket

$$\{k_a, k_b\} = -f_{ab}{}^{c}k_c\,. \tag{5.15}$$

To obtain the answer, we use that the vectors l^{μ}_{a} and r^{μ}_{a} generate the algebra $\mathbf{g}_{-}$,

$$\{l_a, l_b\} = f_{ab}{}^{c}l_c\,, \qquad \{r_a, r_b\} = -f_{ab}{}^{c}r_c\,, \qquad \{l_a, r_b\} = 0\,. \tag{5.16}$$

We find that k^{μ}_{a} satisfies (5.15) if and only if the following holds,

$$f_{ab}{}^{s}R^{l}{}_{s} - f_{mn}{}^{l}R^{m}{}_{a}R^{n}{}_{b} = -\mathcal{L}_{[a}R^{l}{}_{b]}\,. \tag{5.17}$$

This is precisely the condition (5.9).

If $R^{l}{}_{b}$ is constant then we find that it corresponds to an automorphism of the Lie algebra $\mathbf{g}_{-}$. In this case k^{μ}_{a} has a clear geometrical interpretation: it generates a twisted adjoint action of the group $\mathcal{G}$ [5].

However, we are interested in the general case of nonconstant $R^{l}{}_{b}$. It is clear from the analysis above that this generalisation occurs in a controlled way. It simply modifies the vector k^{μ}_{a} by allowing X-dependence of the map $R^{l}{}_{b}$ in the definition of k^{μ}_{a}. Moreover, the deviation of $R^{l}{}_{b}$ from being an automorphism is governed by (5.17).

In conclusion, we see that the most general $N = 2$ superconformal boundary conditions are a very precise generalisation of the well-known ones that correspond to constant Lie algebra automorphisms. In geometrical terms, the resulting D-branes are related in a specific way to the known D-branes that correspond to conjugacy classes of $\mathcal{G}$.

It is interesting to examine the integrability conditions for eqs. (5.9), (5.10), (5.13) and (5.14). We focus on (5.9), hit it on both sides with $\mathcal{L}_c$ and antisymmetrise in a, b and c, to get

$$f_{[ab}{}^{d}\mathcal{L}_{c]}R^{l}{}_{d} - f_{mn}{}^{l}\mathcal{L}_{[c}\left(R^{m}{}_{a}R^{n}{}_{b]}\right) = -2\mathcal{L}_{[c}\mathcal{L}_{a}R^{l}{}_{b]}\,. \tag{5.18}$$

Using that $\mathcal{L}_{[a}\mathcal{L}_{b]} = -f_{ab}{}^{d}\mathcal{L}_{d}$ (which follows from (5.15)), we then rewrite the right-hand side of (5.18) as

$$f_{[ab}{}^{d}\mathcal{L}_{|d|}R^{l}{}_{c]}\,, \tag{5.19}$$

which, using (5.9) as well as the Jacobi identity, expands to

$$f_{[ab}{}^{d}\mathcal{L}_{c]}R^{l}{}_{d} + f_{mn}{}^{l}f_{[ab}{}^{d}R^{m}_{\ |d|}R^{n}_{\ c]}\,. \tag{5.20}$$

On the other hand expanding the left-hand side of (5.18), and again using the Jacobi identity, yields

$$f_{[ab}{}^{d}\mathcal{L}_{c]}R^{l}{}_{d} + 2f_{mn}{}^{l}f_{[ab}{}^{d}R^{m}_{\ |d|}R^{n}_{\ c]}\,. \tag{5.21}$$

Thus (5.18) is satisfied if and only if (5.20) and (5.21) are equal, i.e. the condition for (5.9) to be integrable is

$$f_{mn}{}^{l}f_{[ab}{}^{d}R^{m}_{\ |d|}R^{n}_{\ c]} = 0\,. \tag{5.22}$$

This equation arises in Lie algebra cohomology, and is solved by

$$R^{m}_{\ s}f_{mn}{}^{l}R^{n}_{\ p} = f_{sp}{}^{q}T^{l}{}_{q}\,, \tag{5.23}$$

for some $T^{l}{}_{q}$. For simple Lie algebras the corresponding cohomology is trivial and this is the *only* solution to (5.22). If $T^{l}{}_{q} = R^{l}{}_{q}$, then (5.23) implies that $R^{l}{}_{q}$ is a Lie algebra automorphism, and otherwise the difference $f_{ab}{}^{q}(T^{l}{}_{q} - R^{l}{}_{q}) = \mathcal{L}_{[a}R^{l}{}_{b]}$ is a measure of the extent to which it fails to be such an automorphism.

With the appropriate change of indices these results also apply to the remaining equations (5.10), (5.13) and (5.14).

6. Discussion

In this letter we have analysed the local classical superconformal boundary conditions for non-linear sigma models with specific target spaces, namely group manifolds. Information about the local boundary conditions is encoded in the gluing matrix $R^{A}_{B} : \mathcal{G} \to \mathbf{g} \otimes \mathbf{g}^{*}$. Using the special features of WZW models we found the conditions that R^{A}_{B} has to obey to preserve superconformal symmetry on the boundary. These conditions are a set of first-order differential equations for the gluing matrix. In the case of $N = 2$ supersymmetry, these equations become especially interesting, and we also analysed their integrability. The geometrical significance of these results remains to be determined.

Our analysis is entirely classical and the well-known D-branes corresponding to Lie algebra automorphims are special solutions of our conditions.

At the quantum level the WZW models are not rational with respect to (super)conformal symmetry and therefore it is problematic to analyse the (super)conformal boundary conditions by means of a rational CFT. Thus, at present, at the quantum level, a full description of D-branes on group manifolds is still an open problem. It would be interesting to see if there are classical branes with nonconstant R^{A}_{B} which have a quantum counterpart.

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A. $N = 2$ currents for the WZW model

Here we list the components of the $N = 2$ stress tensor $T_{\pm\pm}$, supersymmetry currents $G^1_\pm$ and $G^2_\pm$, and R-symmetry currents $J_\pm$, in terms of the Lie algebra valued affine currents $j^A_\pm$ and $\mathcal{K}^A_{\pm\!\!\pm}$. These are obtained by substituting the relations (4.2), (4.3), (4.8) and (4.9) into the expressions (2.5)–(2.8) and (3.2)–(3.5).

$$T_{++} = \mathcal{K}^A_{+\!\!+}\eta_{AB}\mathcal{K}^B_{+\!\!+} + i j^A_+ \eta_{AB} \partial_{+\!\!+} j^B_+ \,, \tag{A.1}$$

$$T_{--} = \mathcal{K}^A_{=}\eta_{AB}\mathcal{K}^B_{=} + i j^A_- \eta_{AB} \partial_{=} j^B_- \,, \tag{A.2}$$

$$G^1_+ = j^A_+ \eta_{AB} \mathcal{K}^B_{+\!\!+} + \frac{i}{6} j^A_+ j^B_+ j^C_+ f_{ABC} \,, \tag{A.3}$$

$$G^1_- = j^A_- \eta_{AB} \mathcal{K}^B_{=} + \frac{i}{6} j^A_- j^B_- j^C_- f_{ABC} \,, \tag{A.4}$$

$$G^2_+ = j^A_+ J_{AB} \mathcal{K}^B_{+\!\!+} - \frac{i}{6} j^A_+ j^B_+ j^C_+ J^D_{\ A} J^L_{\ B} J^M_{\ C} f_{DLM} \,, \tag{A.5}$$

$$G^2_- = j^A_- J_{AB} \mathcal{K}^B_{=} - \frac{i}{6} j^A_- j^B_- j^C_- J^D_{\ A} J^L_{\ B} J^M_{\ C} f_{DLM} \,, \tag{A.6}$$

$$J_+ = j^A_+ j^B_+ J_{AB} \,, \qquad J_- = j^A_- j^B_- J_{AB} \,. \tag{A.7}$$

References

[1] A.Y. Alekseev and V. Schomerus, *D-branes in the WZW model*, *Phys. Rev.* **D 60** (1999) 061901 [hep-th/9812193].

[2] V. Schomerus, *Lectures on branes in curved backgrounds*, *Class. and Quant. Grav.* **19** (2002) 5781 [hep-th/0209241].

[3] C. Bachas, M.R. Douglas and C. Schweigert, *Flux stabilization of D-branes*, *J. High Energy Phys.* **05** (2000) 048 [hep-th/0003037].

[4] G. Felder, J. Fröhlich, J. Fuchs and C. Schweigert, *The geometry of WZW branes*, *J. Geom. Phys.* **34** (2000) 162 [hep-th/9909030].

[5] S. Stanciu, *D-branes in group manifolds*, *J. High Energy Phys.* **01** (2000) 025 [hep-th/9909163].

[6] A.Y. Alekseev, A. Recknagel and V. Schomerus, *Non-commutative world-volume geometries: branes on* SU(2) *and fuzzy spheres*, *J. High Energy Phys.* **09** (1999) 023 [hep-th/9908040].

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[7] A.Y. Alekseev, A. Recknagel and V. Schomerus, *Brane dynamics in background fluxes and non-commutative geometry*, *J. High Energy Phys.* **05** (2000) 010 [hep-th/0003187].

[8] A.Y. Alekseev, A. Recknagel and V. Schomerus, *Open strings and non-commutative geometry of branes on group manifolds*, *Mod. Phys. Lett.* **A 16** (2001) 325 [hep-th/0104054].

[9] A.Y. Alekseev, S. Fredenhagen, T. Quella and V. Schomerus, *Non-commutative gauge theory of twisted D-branes*, *Nucl. Phys.* **B 646** (2002) 127 [hep-th/0205123].

[10] M.R. Gaberdiel, A. Recknagel and G.M.T. Watts, *The conformal boundary states for* SU(2) *at level 1*, *Nucl. Phys.* **B 626** (2002) 344 [hep-th/0108102].

[11] C. Albertsson, U. Lindström and M. Zabzine, $N = 1$ *supersymmetric sigma model with boundaries, I*, *Commun. Math. Phys.* **233** (2003) 403 [hep-th/0111161].

[12] C. Albertsson, U. Lindström and M. Zabzine, $N = 1$ *supersymmetric sigma model with boundaries, II*, hep-th/0202069.

[13] U. Lindström and M. Zabzine, $N = 2$ *boundary conditions for non-linear sigma models and Landau-Ginzburg models*, *J. High Energy Phys.* **02** (2003) 006 [hep-th/0209098].

[14] U. Lindström and M. Zabzine, *D-branes in* $N = 2$ *WZW models*, *Phys. Lett.* **B 560** (2003) 108 [hep-th/0212042].

[15] J. Gates, S. J., C.M. Hull and M. Roček, *Twisted multiplets and new supersymmetric nonlinear sigma models*, *Nucl. Phys.* **B 248** (1984) 157.

[16] S. Lyakhovich and M. Zabzine, *Poisson geometry of sigma models with extended supersymmetry*, *Phys. Lett.* **B 548** (2002) 243 [hep-th/0210043].

[17] P. Spindel, A. Sevrin, W. Troost and A. Van Proeyen, *Complex structures on parallelized group manifolds and supersymmetric sigma models*, *Phys. Lett.* **B 206** (1988) 71.

[18] P. Spindel, A. Sevrin, W. Troost and A. Van Proeyen, *Extended supersymmetric sigma models on group manifolds, 1. The complex structures*, *Nucl. Phys.* **B 308** (1988) 662.

[19] A. Sevrin, W. Troost, A. Van Proeyen and P. Spindel, *Extended supersymmetric sigma models on group manifolds, 2. Current algebras*, *Nucl. Phys.* **B 311** (1988) 465.

[20] V.G. Drinfeld, *Quantum groups*, *J. Sov. Math.* **41** (1988) 898.

[21] S.E. Parkhomenko, *Extended superconformal current algebras and finite dimensional manin triples*, *Sov. Phys. JETP* **75** (1992) 1.